Acceleration of the Remez Exchange Algorithm for the Design of L_∞ Optimum FIR Filters

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ABSTRACT

In this paper a new initialization scheme for the Remez exchange algorithm is proposed. More specifically, the solution of the well known "don't care" filter design method is proposed as a new efficient initialization scheme for the Remez algorithm. Our proposal is motivated by the fact that the "don't care" least squares optimum solution satisfies the one of the two basic conditions that are sufficient for obtaining the L_∞ optimum solution according to the Alternation Theorem and at the same time it adequately approximates the second one. Because of these properties we have a significant speed up of the convergence of the Remez exchange algorithm to the L_∞ optimum solution.

1 Introduction

In FIR filter design theory, the L_∞ norm is a very widely used approximation measure. Approximations obtained by this criterion, also known as Chebyshev or Min-max, exhibit equiripple behavior in the frequency bands of interest (passband and stopband). This type of behavior is clearly very desirable in practice because it completely eliminates, in an optimum way, the Gibb's phenomenon observed in other (especially L_2) approximation techniques.

Obtaining the optimum L_∞ solution is quite a difficult task. All existing algorithms that obtain the Chebyshev approximations are iterative and require an increased complexity. Most well known such algorithms are the Remez Exchange Algorithm (REA), the Iterative weighted least squares and algorithms based on Constraint linear optimization. In this work we will concentrate on the REA [6], [7], [12].

In most iterative techniques the time required by the algorithm to converge depends closely on the initial "guess" of the solution. The "better" this selection is, the faster the algorithm converges to the final solution. It is also not uncommon, bad initial selections, to lead to divergence. The first initialization method for REA was presented in [7] and additional methods aiming in speeding up the algorithm in [1], [4]. In this paper we propose an alternative initialization scheme. Specifically we propose the use of the "don't care" least squares optimum solution [11, page 70] as a starting point for the REA. This proposal is strongly supported by a theoretical result stating that the "don't care" solution has the right number of alternating in sign extrema, required by the Alternation Theorem, inside the bands of interest. Thus it satisfies exactly one of the two conditions that define the optimum L_∞ solution. On the other hand the "don't care" method is known to yield very good (in the L_∞ sense) filters that approximate sufficiently close the second condition that defines the optimum L_∞ filter.

2 L_∞ Approximations and the Alternation Theorem

Let us first define the approximation problem we are interested in. Consider a collection of a finite number of closed nonoverlapping intervals I_i, $i = 1, \ldots, K$ that are subsets of $[0\ \pi]$. Consider also a real function $D(\omega)$, continuous and known on each interval I_i, which we like to approximate in the L_∞ sense. Let $\phi_n(\omega) = \cos(n\omega)$, or $\phi_n(\omega) = \sin((n+1)\omega)$, $n = 0, \ldots, N-1$ the two sets of base function we like to use to approximate the function $D(\omega)$. Finally let $W(\omega)$ be a weighting function which is known, continuous and nonnegative on each interval I_i.

The correspondence with the filter design problem is obvious. The intervals I_i constitute either the passbands or the stopbands, $D(\omega)$ is the desired response and $W(\omega)$ is the weight in each band. The first set of base functions can be used to approximate even symmetric desired responses (defined on $[-\pi\ \pi]$) while the second odd symmetric. It is also clear that the open intervals between consecutive I_i constitute the transition regions between the bands of interest.

Let $h = [h_0 h_1 \ldots h_{N-1}]^t$ be a vector of coefficients and denote by $H(h, \omega) = \sum_{n=0}^{N-1} h_n \phi_n(\omega)$ a linear combination of the base functions. We are interested in obtaining the optimum h_∞ in the L_∞ sense, that satisfies

$$h_o = \arg\left\{ \inf_{h} \sup_{\omega \in \cup_{i=1}^{b} I_i} |W(\omega)[D(\omega) - H(h,\omega)]| \right\} \qquad (1)$$

We have now the following theorem that gives necessary and sufficient conditions for h_o.

Theorem 1: The vector h_o is the optimum in the L_∞ sense if and only if the following two conditions are satisfied:

i). The weighted error function $W(\omega)[D(\omega) - H(h_o,\omega)]$ has at least $N+1$ local extrema with alternating sign at points $\omega_1 < \omega_2 < \cdots < \omega_{N+1}$ that belong to the bands of interest (i.e. $\omega_n \in \cup_{i=1}^{b} I_i$).

ii). The local extrema of condition *i)* are all equal, in absolute value, to the maximum weighted absolute deviation $\sup_{\omega \in \cup_{i=1}^{b} I_i} |W(\omega)[D(\omega) - H(h_o,\omega)]|$.

Proof: The proof can be found in [8],[13]. ■

The above theorem is also known as the *Alternation Theorem* and completely characterizes the optimum solution h_o. From *i)* and *ii)* we can conclude that a vector h can be regarded as a good initial guess for any recursive algorithm that tries to estimate h_o, if it satisfies condition *i)* exactly and condition *ii)* approximately. Notice that the main difficulty for any such initialization scheme is to insure the existence of at least $N+1$ local extrema, with alternating sign, INSIDE the set $\cup_{i=1}^{b} I_i$.

3 "Don't Care" Optimum Least Squares Approximations

Let $D(\omega)$, $W(\omega)$, $\phi_n(\omega)$ be as in the previous section. Recall that $W(\omega)$ was defined only on the set $\cup_{i=1}^{b} I_i$. If we extend $W(\omega)$ to the whole interval $[0,\pi]$, by setting $W(\omega) = 0$ for $\omega \in [0,\pi] - \cup_{i=1}^{b} I_i$ we can then define a vector of optimum coefficients h_d by solving the following least squares problem

$$h_d = \arg\left\{ \inf_{h} \int_0^{\pi} W^2(\omega)[D(\omega) - H(h,\omega)]^2 d\omega \right\} \qquad (2)$$

Since the weighting function $W(\omega)$ is zero outside the set of interest $\cup_{i=1}^{b} I_i$ the values of $D(\omega)$ outside $\cup_{i=1}^{b} I_i$ play absolutely no role (this is why the term "don't care" is used). The optimum vector h_d is the solution to the linear system defined by the equations

$$\int_0^{\pi} W^2(\omega)[D(\omega) - H(h_d,\omega)]\phi_l(\omega)d\omega = 0 \qquad (3)$$

with $l = 0, \ldots, N-1$.

It is easy to show that the linear system obtained by (3) has a Toeplitz plus Hankel structure. This allows for the use of specialized algorithms, with reduced complexity [9], for finding the solution. It is also known that the "don't care" method yields very satisfactory solutions [2],[3], in the sense that the maximum ripple is only a few dB larger than the optimum L_∞ ripple, thus approximating condition *ii)*. In order to prove that h_d generates the correct number of local extrema with alternating sign inside the bands of interest, we are going to use a theorem due to Motzkin and Walsh from their work in the area of the Approximation Theory [10]. The same theorem was used by Rice and Usow in [14] in order to theoretically show the convergence of the Lawson algorithm [5].

Theorem 2: Let $\{\phi_n(\omega)\}$ be a Chebyshev set and let $H(h,\omega) = \sum_{n=0}^{N-1} h_n \phi_n(\omega)$ be a linear combination of this set, where h denotes the parameter vector $[h_0 h_1 \ldots h_{N-1}]$. Then, if $H(h^*,\omega)$ is a best weighted L_p, $(p < 1 < \infty)$ approximation to a continuous function $f(\omega)$ on a set X which is composed by a number of nonoverlapping intervals of $[0,\pi]$, $W(\omega)H(h^*,\omega)$ strongly interpolates $W(\omega)f(\omega)$ inside the set X.

Proof: The proof can be found in [10]. ■

Notice that $H(h_d,\omega)$, with h_d defined as in (2) satisfies all the requirements of Theorem 2. Taking into account that $H(h,\omega)$ is said to strongly interpolate $f(\omega)$ N times if

$$(-1)^i[H(h,\omega_i) - f(\omega_i)] \geq 0 \qquad (4)$$

for some $N+1$ points ω_i in the set X, it is clear that the "don't care" least squares optimum solution has a weighted error that achieves at least $N+1$ local extrema with alternating sign inside the set $\cup_{i=1}^{b} I_i$. In other words it satisfies exactly condition *i)* of Theorem 1.

Summarizing, we have that the "don't care" least squares optimum solution h_d is first of all easily computable because of its special Toeplitz plus Hankel structure, second it satisfies exactly the most difficult condition of the Alternation Theorem and third from practice it is known that it approximates the second condition of the Alternation Theorem. It can thus be considered as a good candidate for initializing the REA.

4 Examples

By modifying the Matlab function *REMEZ* to accept our initialization scheme we designed a large number of lowpass and bandpass filters with varying weights in each band. We considered filter lengths ranging from 21 up to 201. In all cases the REA, with the new initialization scheme, required from 10% to 85% less iterations to converge as compared to the classical initialization scheme. Some results obtained from the application of the proposed initialization scheme are contained in Table I and Table II. Specifically, in Table I we present the number of iterations needed by the REA to converge under the two initialization schemes for different specifications of a lowpass filter. The width of the transition band of the filters, in all cases, was 0.1. From the last column of Table I we can easily conclude that with the proposed scheme we have a significant speed up of REA. Table II contains the results obtained from the design of a number of bandpass filters. For all cases the two transition bands had width equal to 0.1. Again for this type of filters the proposed initialization scheme significantly improves the convergence speed of the REA.

Filter Specs		Exist.	Prop.	Saving
$2N+1$	ω_p	Iter.	Iter.	%
101	0.1	9	4	55.6
	0.2	8	4	50.0
	0.3	14	4	71.4
	0.4	10	6	40.0
	0.5	9	5	44.4
121	0.1	12	5	58.3
	0.2	10	4	60.0
	0.3	8	4	50.0
	0.4	9	4	55.6
	0.5	9	4	55.6
141	0.1	12	5	58.3
	0.2	10	4	60.0
	0.3	10	5	50.0
	0.4	10	6	40.0
	0.5	10	5	50.0
161	0.1	13	5	61.5
	0.2	10	4	60.0
	0.3	10	4	60.0
	0.4	11	4	63.6
	0.5	12	4	66.7
181	0.1	13	4	69.2
	0.2	11	4	63.6
	0.3	11	5	54.5
	0.4	10	6	40.0
	0.5	11	5	54.5
201	0.1	16	5	68.8
	0.2	13	5	61.5
	0.3	14	4	71.4
	0.4	13	4	69.2
	0.5	11	4	63.6

Table I. Number of iterations required by the REA to converge under the proposed and the existing initialization scheme for different lowpass filter specifications.

5 Conclusion

We have shown that the "don't care" optimum solution guarantees the existence of the necessary number of, alternating in sign, extrema required by the Alternation Theorem, thus satisfying one of the two basic conditions of this theorem for the determination of the L_∞ optimum solution while at the same time it approximates sufficiently close the second condition. Because of these properties the "don't care" least squares optimum solution, if used as an initialization scheme for the Remez exchange algorithm, results in a significant increase of its convergence speed as it was demonstrated by a number of filter design examples.

Filter Specs			Exist.	Prop.	Saving
$2N+1$	ω_p	ω_s	Iter.	Iter.	%
101	0.1	0.3	11	6	45.5
	0.2	0.4	14	7	50.0
	0.3	0.5	26	6	76.9
	0.4	0.6	14	5	64.3
121	0.1	0.3	15	4	73.3
	0.2	0.4	14	5	64.3
	0.3	0.5	17	6	64.7
	0.4	0.6	16	5	68.8
141	0.1	0.3	21	9	57.1
	0.2	0.4	21	10	52.4
	0.3	0.5	22	9	59.1
	0.4	0.6	20	7	65.0
161	0.1	0.3	20	5	75.0
	0.2	0.4	21	6	71.4
	0.3	0.5	23	5	78.3
	0.4	.6	23	5	78.3
181	0.1	0.3	17	4	76.5
	0.2	0.4	16	7	56.3
	0.3	0.5	19	8	57.9
	0.4	0.6	22	6	72.7
201	0.1	0.3	20	5	75.0
	0.2	0.4	20	5	75.0
	0.3	0.5	18	6	66.7
	0.4	0.6	21	6	71.4

Table II. Number of iterations required by the REA to converge under the proposed and the existing initialization scheme for different bandpass filter specifications.

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